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QUIZ 8

Space, subspace: Let $V_n = \{0,1\}^n$, $N = |V_n| = 2^n$. We add and subtract elements in V_n component-wise modulo 2. So 0^n is the *identity* for addition, and $x \in V_n$ is its own additive inverse (because $x + x = 0^n$). A subset $S \subset V_n$ is a subspace (or subgroup) of V_n if it is closed under addition.

Question 1: Which of the following are subspaces of V_4?

$$S_1 = \{0000, 0110\} \qquad S_2 = \{0000, 0100, 0010, 1100\}$$
$$S_3 = \{0000\} \qquad S_4 = \{0000, 0001, 1000, 1001, 0110, 0111, 1110, 1111\}$$

Perpendicular subspace: For $x, y \in V_n$, let

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \pmod 2.$$

For a subspace S of V_n, let

$$S^{\perp} = \{y \in V_n : x \cdot y = 0 \text{ for all } x \in S\}.$$

Question 2: What is $S_3^{\perp}$?

The Hadamard transform and subspaces: For a subspace S of V_n, let

$$|S\rangle = \frac{1}{\sqrt{|S|}} \sum_{x \in \{0,1\}^n} |x\rangle.$$

Recall that

$$H^{\otimes n} : |x\rangle \to \frac{1}{\sqrt{N}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle.$$

Question 3: Compute $H^{\otimes 4} |S_1\rangle$.